ECS 452: Digital Communication Systems **2013/1**

HW 3 — Due: Sep ~~17~~ 18

Lecturer: Prapun Suksompong, Ph.D.

Instructions

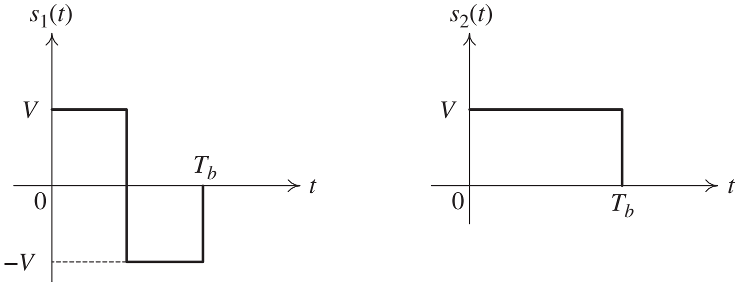
(a) ONE part of a question will be graded (5 pt). Of course, you do not know which part will be selected; so you should work on all of them.

(b) It is important that you try to solve all problems. (5 pt)

(c) Late submission will be heavily penalized.

(d) Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

Problem 1. Consider two waveforms $s_1(t)$ and $s_2(t)$ shown in Figure 3.1. A communication system uses these two waveforms to transmit each bit of information. To transmit bit 0, waveform $s_1(t)$ is transmitted. To transmit bit 1, waveform $s_2(t)$ is transmitted. The channel is assumed to be additive white Gaussian noise with PSD $\frac{N_0}{2}$. Bits 0 and 1 are also assumed to be chosen with equal probability. The detector used at the receiving end is the optimal detector.



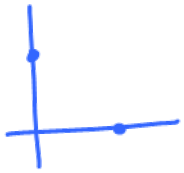


Figure 3.1: Signal set for Question 1

(a) Find the probability of detection error for this system. (You may use the constellation derived in an earlier assignment.)

(b) Plot $P(\mathcal{E})$ (in log scale) vs. E_b/N_0 (in dB). Use `MATLAB` to compare the theoretical results and simulation results.

(c) Draw the block diagram of the receiver that implements optimal detection with matched filters.

Problem 2. Consider a ternary constellation. Assume that the three vectors are equiprobable.

(a) Suppose the three vectors are

$$\boldsymbol{s}^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \boldsymbol{s}^{(2)} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \text{ and } \boldsymbol{s}^{(3)} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

Find the corresponding average energy per symbol.

(b) Suppose we can shift the above constellation to other location; that is, suppose that the three vectors in the constellation are

$$\boldsymbol{s}^{(1)} = \begin{pmatrix} 0 - a_1 \\ 0 - a_2 \end{pmatrix}, \boldsymbol{s}^{(2)} = \begin{pmatrix} 3 - a_1 \\ 0 - a_2 \end{pmatrix}, \text{ and } \boldsymbol{s}^{(3)} = \begin{pmatrix} 3 - a_1 \\ 3 - a_2 \end{pmatrix}.$$

Find a_1 and a_2 such that corresponding average energy per symbol is minimum.

Problem 3. Consider four rectangular 24-ary schemes.

(a) 1×24 constellation

(b) 2×12 constellation

(c) 3×8 constellation

(d) 4×6 constellation

Each scheme are derived from the waveform models whose noise process is additive white Gaussian noise with PSD $N_0/2$. All points are equally likely to be transmitted. Each constellation is centered at the origin (so that the average E_s is minimized.) Let d be the vertical distances and horizontal distances between any adjacent points. Optimal detection is used.

The probability of error for each of the constellation is of the form

$$P(\mathcal{E}) = Aq + Bq^2$$

where

$$q = Q\left(\sqrt{C \times \frac{E_b}{N_0}}\right).$$

(i) Find the constants A, B, and C for each of the cases.

(ii) Compare the performance of these four schemes.

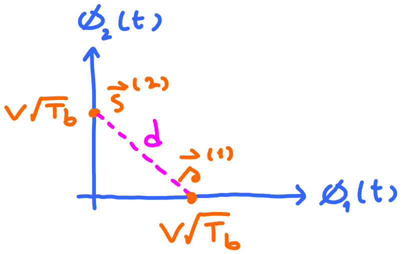
(iii) Compare the theoretical results with `MATLAB` simulations.

Q1 Binary Orthogonal Scheme

Wednesday, September 11, 2013
10:07 AM

(a)

From HW 1, we have already used GSOP to derive the constellation :



Remark: we know that, for equiprobable binary signalling schemes,

$$P(\mathcal{E}) = Q\left(\frac{d}{2\sigma}\right).$$

The distance between the two points is $d \overset{\text{Pythagorus' theorem}}{=} \sqrt{2}\,V\sqrt{T_b}$.
So, the corresponding probability of (decoding) error is

$$P(\mathcal{E}) = Q\left(\frac{d}{2\sigma}\right) = Q\left(\frac{V\sqrt{2T_b}}{2\sigma}\right) = Q\left(\frac{V\sqrt{T_b}}{\sqrt{2}\,\sigma}\right)$$

For AWGN channel with PSD $\frac{N_0}{2}$, we have $\sigma^2 = \frac{N_0}{2}$.

Therefore, $P(\mathcal{E}) = Q\left(\frac{V\sqrt{T_b}}{\sqrt{N_0}}\right) = Q\left(V\sqrt{\frac{T_b}{N_0}}\right)$.

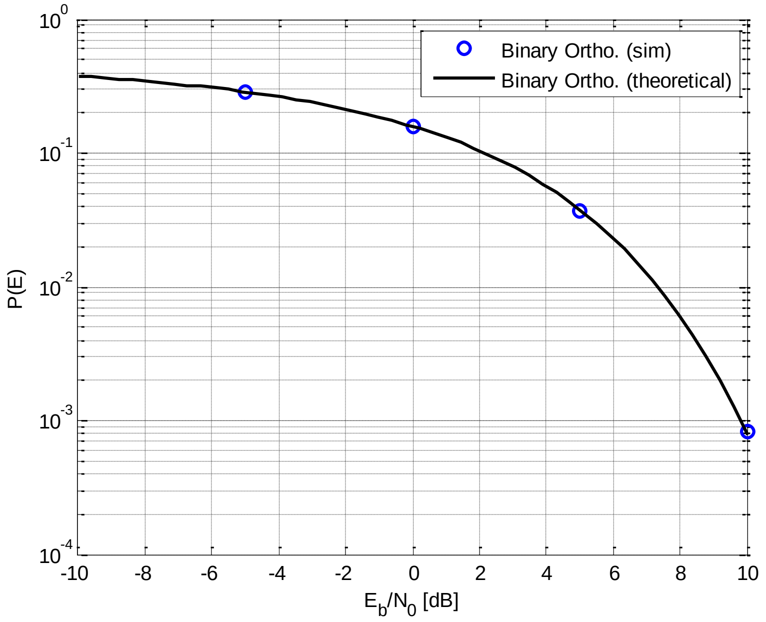
(b) The energy of both points is V^2T_b. So, $E_s = V^2T_b$.

Here, $M = 2$. So, $E_b = \frac{E_s}{\log_2 2} = V^2T_b$.

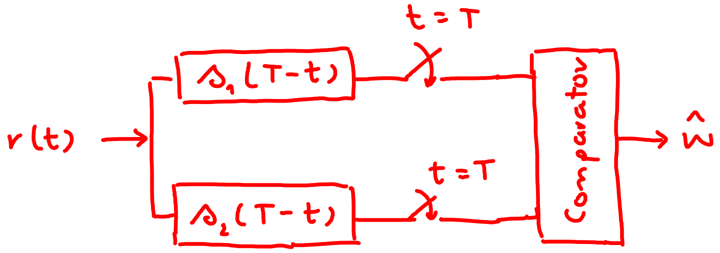
Rewriting $P(\mathcal{E})$ in part (a), we have

$$P(\mathcal{E}) = Q\left(\sqrt{\frac{V^2T_b}{N_0}}\right) = \underbrace{Q\left(\sqrt{\frac{E_b}{N_0}}\right)}.$$

In fact, we can answer this directly by realizing that this is an equiprobable binary orthogonal signaling scheme.



(c) The optimal detector can be implemented from matched filter as followed

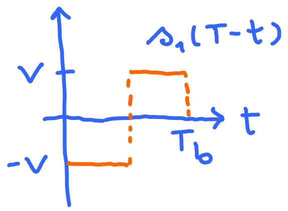


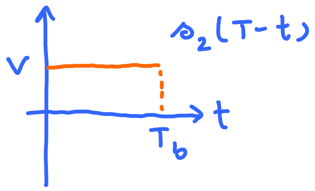
Note that

① There is no need for the bias term because the waveforms are equally likely and of equal energy.

② To make matched filter causal, we choose $T \geq T_b$.
To minimize delay, we choose $T = T_b$.
In which case, the plots for $s_1(T-t)$ and $s_2(T-t)$ are shown below:

$$\left(s_i(T-t) = s_i\big(-(t-T)\big). \right)$$

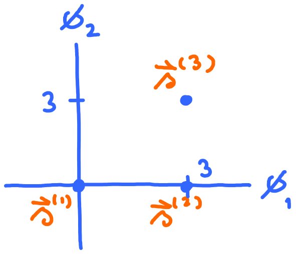




Q2 Minimum Energy for Constellation

Thursday, September 05, 2013
3:31 PM

(a)



$$\vec{s}^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\vec{s}^{(2)} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$\vec{s}^{(3)} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$$E_s = \frac{1}{3}\left(0^2 + 3^2 + 3^2 + 3^2\right) = 9$$

(b) $$E_s = \sum_{i=1}^{3} p_i \sum_{j=1}^{2} \left(\vec{s}_j^{(i)}\right)^2 = \frac{1}{3}\left(\sum_{i=1}^{3}\left(\vec{s}_1^{(i)}\right)^2 + \sum_{i=1}^{3}\left(\vec{s}_2^{(i)}\right)^2\right)$$

$$= \frac{1}{3}\left((0-a_1)^2 + (2-a_1)^2 + (2-a_1)^2\right) + \frac{1}{3}\left((0-a_2)^2 + (0-a_2)^2 + (2-a_2^2)\right)$$

In general, we have to minimize terms of the form

$$\sum_i p_i (x_i - a)^2 = \mathbb{E}\left[(X-a)^2\right] = \mathbb{E}\left[(X - \underbrace{-\mathbb{E}X + \mathbb{E}X}_{0} - a)^2\right]$$

$$= \operatorname{Var} X + 2\,\underbrace{\mathbb{E}\left[(X - \mathbb{E}X)\right]}_{0}(\mathbb{E}X - a) + (\mathbb{E}X - a)^2$$

$$= \operatorname{Var} X + \underbrace{(\mathbb{E}X - a)^2}_{\uparrow}$$

the only term that depends on a.
Minimum value of 0 is achieved
when $a = \mathbb{E}X$.

So, the minimum value occurs when $a = \mathbb{E}X = \sum_i p_i x_i$

Minimum E_s occurs when

$$a_1 = \frac{1}{3}(0+3+3) = 2. \qquad a_2 = \frac{1}{3}(0+0+3) = 1.$$

Q3 24-QAM

Wednesday, September 11, 2013
10:23 AM

① For 1-D standard rectangular M-PAM

$$P(\mathcal{E} \mid \vec{s} = \vec{s}^{(i)}) = \begin{cases} Q\left(\frac{d}{2\sigma}\right), & i = 1, M \\ 2Q\left(\frac{d}{2\sigma}\right), & i = 2, 3, \dots, M-1 \end{cases}$$

$$\text{So, } P(\mathcal{E}) = \frac{1}{M}\left(2 \times q + (M-2)2q\right) = \frac{1}{M}\left((M-1)2q\right) = 2\frac{M-1}{M} q = 2\frac{M-1}{M} Q\left(\frac{d}{2\sigma}\right)$$

$$\Downarrow$$

$$A = 2\frac{M-1}{M} \overset{M=24}{=} \frac{23}{12}$$

$$B = 0$$

For 2-D standard rectangular M-QAM

To find $P(\mathcal{E})$, recall that we have three cases of points

Case	~~#~~	$P(\mathcal{E} \mid \vec{s} = \vec{s}^{(i)})$
corner	n_1	$2q - q^2$
middle	n_2	$3q - 2q^2$
center	n_3	$4q - 4q^2$

$\Big\}\ q = Q\left(\frac{d}{2\sigma}\right)$

Therefore, we simply count the ~~#~~ points in each case and use those numbers as weights for $P(\mathcal{E})$:

$$P(\mathcal{E}) = \frac{1}{M}\left(n_1(2q - q^2) + n_2(3q - 2q^2) + n_3(4q - 4q^2)\right)$$

$$\Rightarrow A = \frac{1}{M}\left(2n_1 + 3n_2 + 4n_3\right)$$

$$\Rightarrow B = -\frac{1}{M}\left(n_1 + 2n_2 + 4n_3\right)$$

$M = M_1 \times M_2$	$n_1 = 4$	$n_2 = 2(M_1 - 2) + 2(M_2 - 2)$	$n_3 = M - n_1 - n_2$	A	B
2×12	4	20	0	$68/24 = 17/6$	$-44/24 = 11/6$
3×8	4	14	6	$74/24 = 37/12$	$-56/24 = 7/3$
4×6	4	12	8	$76/24 = 19/6$	$-60/24 = 5/2$

② To change $\frac{d}{2\sigma}$ to $\frac{E_b}{N_0}$, we need to find E_b.

To do this, we start with E_s ← average energy per symbol.

Remark: There are many ways to find E_s. As long as you can find the coordinates of the points in the constellation, then, it is straightforward to find the energy of each point and then average all the energy. This can be done easily in MATLAB. However, here, we show how to derive the answer analytically.

Standard 1-D M-PAM:

Start with $(1, 2, 3, \dots, M) \times d$

↑ this changes the spacing btw the pts to d.

Then, we shift the constellation so that the center is at origin. To do this, we simply subtract the average out.

at origin. To do this, we simply subtract the average out.

$$(1, 2, 3, \dots, M) \times d \; - \underbrace{\frac{1}{M}\left(\sum_{k=1}^{M} k\right) d}_{\substack{\uparrow \\ \text{call this as } m \,=\, \frac{1}{\cancel{M}} \frac{(M+1)\cancel{M}}{2} d}}$$

Viewing this as a RV, we may think about

RV $U \sim$ uniform on $1, 2, \dots, M$

RV $S = dU - \mathbb{E}[dU] = d(U - \mathbb{E}U)$

So, $\mathbb{E}S = 0$ and $\mathbb{E}[S^2] = \text{Var}\, S = d^2 \, \text{Var}\, U$

Next, we find the average energy:

$$E_s = \frac{1}{M}\left(\sum_{k=1}^{M} (kd - m)^2\right) = \frac{1}{M}\left(\sum_{k=1}^{M} k^2 d^2 \underbrace{- 2md \overbrace{\sum_{k=1}^{M} k}^{M \times m}}_{-2Mm^2} + m^2 M\right)$$

$$= \frac{1}{M}\left(\sum_{k=1}^{M} k^2 d^2 - M m^2\right)$$

$$= d^2\left(\left(\frac{1}{M}\sum_{k=1}^{M} k^2\right) - \left(\frac{1}{M}\sum_{k=1}^{M} k\right)^2\right) = d^2\left(\frac{1}{3}(M^2-1) - \left(\frac{M+1}{2}\right)^2\right)$$

$$= d^2\left((M+1)\left(\frac{M-1}{3} - \frac{M+1}{4}\right)\right) = d^2 \frac{(M+1)(M-1)}{12} = \frac{1}{12} d^2 (M^2-1)$$

Some facts about summation:

$$\sum_{k=1}^{M} k = \frac{M(M+1)}{2}$$

$$\begin{array}{rl} A = & 1 + 2 + \cdots + M \\ A = & M + M-1 + \cdots + 1 \quad + \\ \hline 2A = & \underbrace{(M+1) + (M+1) + \cdots + (M+1)}_{M \text{ terms}} \end{array}$$

$$\sum_{k=1}^{M} k^2 = \sum_{k=1}^{M} k(k-1) + \sum_{k=1}^{M} k$$

$$\sum_{k=1}^{M} k(k-1) = 1\cdot 0 + 2\cdot 1 + 3\cdot 2 + \cdots + M(M-1)$$

$$= 1\cdot 0\cdot 3 + 2\cdot 1\cdot 3 + 3\cdot 2\cdot 3 + \cdots + M(M-1)\cdot 3$$

$$= 1\cdot 0 \cdot \frac{2-(-1)}{3} + 2\cdot 1 \cdot \frac{3-0}{3} + \cdots + M(M-1)\frac{(M+1)-(M-2)}{3}$$

$$= \frac{1}{3}\Big(\cancel{(2\cdot 1\cdot 0)} - (1\cdot 0\cdot(-1)) + \cancel{(3\cdot 2\cdot 1)} - \cancel{(2\cdot 1\cdot 0)} +$$

$$\cdots + (M+1)(M)(M-1) - \cancel{M(M-1)(M-2)}\Big)$$

$$= \frac{1}{3} M(M+1)(M-1)$$

$$\sum_{k=1}^{M} k^2 = \frac{1}{3} M(M+1)(M-1) + \frac{M(M+1)}{2} = M(M+1)\left(\frac{M-1}{3} + \frac{1}{2}\right)$$

$$= \frac{1}{6} M(M+1)(2M+1)$$

Alternatively, we have $E_s = \mathbb{E}[S^2] = d^2 \, \text{Var}\, U = d^2\left(\left(\frac{1}{M}\sum_{k=1}^{M} k^2\right) - \left(\frac{1}{M}\sum_{k=1}^{M} k\right)^2\right)$ ← same as above.

$$= d^2\left(\frac{M^2-1}{12}\right)$$

Knowing E_s, we can then find $E_b = E_s / \log_2 M = \frac{1}{12} \frac{d^2 (M^2-1)}{\log_2 M}$

$$d^2 = \frac{12 (\log_2 M) E_b}{}$$

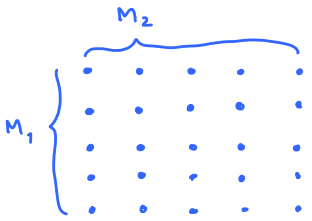
M^2-1

$$\frac{d}{2\sigma} = \sqrt{\frac{d^2}{2N_0}} = \sqrt{\underbrace{\frac{6}{M^2-1}(\log_2 M)}\frac{E_b}{N_0}}$$

$$C = \frac{6}{M^2-1}(\log_2 M) \overset{\substack{M=24\\ \downarrow}}{=} \frac{6}{575}\log_2 24$$

Now for 2-D standard M-ary QAM.

Suppose $M = M_1 \times M_2$.



$$E_s = \mathbb{E}\left[\|\vec{s}\|^2\right] = \mathbb{E}\left[(s_1)^2 + (s_2)^2\right] = \mathbb{E}\left[s_1^2\right] + \mathbb{E}\left[s_2^2\right]$$

s_1: the first component of $\vec{s}$ — s_2: the second component of $\vec{s}$

Let $U_j \sim$ Uniform on $1:M_j$.

Define $S_j = dU_j - \mathbb{E}[dU_j] = d(U_j - \mathbb{E}U_j)$

Then, $\mathbb{E}[s_1^2] = d^2 \operatorname{Var} U_j = d^2 \frac{M_j^2-1}{12}$

Therefore, $E_s = \frac{d^2}{12}\left(M_1^2+M_2^2-2\right)$

$$E_b = \frac{d^2}{12}\,\frac{M_1^2+M_2^2-2}{\log_2 M} \Rightarrow d^2 = \frac{12}{M_1^2+M_2^2-2}\log_2 M \;\; E_b$$

$$\frac{d}{2\sigma} = \sqrt{\frac{d^2}{4\sigma^2}} = \sqrt{\frac{d^2}{2N_0}} = \sqrt{\underbrace{\frac{6}{M_1^2+M_2^2-2}(\log_2 M)}\frac{E_b}{N_0}}$$

$$C = \frac{6}{M_1^2+M_2^2-2}\log_2 M$$

(i) Summary:

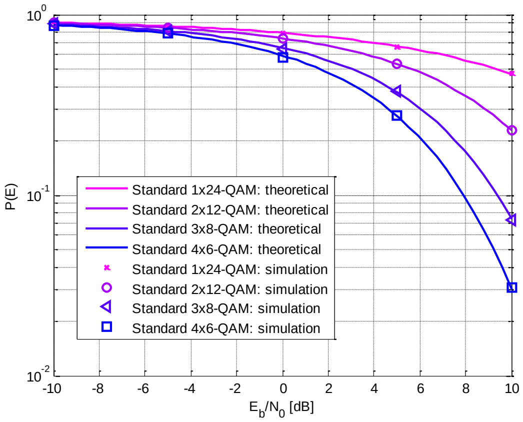
$M=M_1\times M_2$	A	B	C
1×24	$\frac{23}{12}$	0	$\frac{6}{575}\times\log_2 24$
2×12	$\frac{17}{6}$	$-\frac{11}{6}$	$\frac{3}{73}\times\log_2 24$
3×8	$\frac{37}{12}$	$-\frac{7}{3}$	$\frac{6}{71}\times\log_2 24$
4×6	$\frac{19}{6}$	$-\frac{5}{2}$	$\frac{3}{25}\times\log_2 24$

(ii) For small $\frac{E_b}{N_0}$ (when $\frac{E_b}{N_0} \to 0$),

$q = Q\left(\sqrt{c\frac{E_b}{N_0}}\right) \to Q(0) = 0.5$

$P(\mathcal{E}) \to \frac{A}{2} + \frac{B}{4} = \frac{23}{24} \approx 0.9583$ (same for all constellation)

For larger $\frac{E_b}{N_0}$, the plots of $P(\mathcal{E})$ shows that the performance is better when the constellation is closer to being a square.



Intuitively, we note that $P(\mathcal{E})$ depends strongly on the distances btw the points in the constellation. The "square" constellation uses less average energy because the points are closer to the origin. So, for a given average energy, the "square" constellation enjoys greater distances btw its point and hence better $P(\mathcal{E})$.

(iii) See the plots in part (ii).

ECS 452: Digital Communication Systems **2013/1**

HW 4 — Due: Oct 4

Lecturer: Prapun Suksompong, Ph.D.

Instructions

(a) ONE part of a question will be graded (5 pt). Of course, you do not know which part will be selected; so you should work on all of them.

(b) It is important that you try to solve all problems. (5 pt)

(c) Late submission will be heavily penalized.

(d) Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

Problem 1. Consider random variables X and Y whose joint pmf is given by

$$p_{X,Y}(x,y) = \begin{cases} c\,(x+y), & x \in \{1,3\} \text{ and } y \in \{2,4\}, \\ 0, & \text{otherwise.} \end{cases}$$

Evaluate the following quantities.

(a) $H(X,Y)$

(b) $H(X)$

(c) $H(Y)$

(d) $H(X|Y)$

(e) $H(Y|X)$

(f) $I(X;Y)$

Problem 2. Consider a pair of random variables X and Y whose joint pmf is given by

$$p_{X,Y}(x,y) = \begin{cases} 1/15, & x = 3, y = 1, \\ 2/15, & x = 4, y = 1, \\ 4/15, & x = 3, y = 3, \\ \beta, & x = 4, y = 3, \\ 0, & \text{otherwise.} \end{cases}$$

Evaluate the following quantities.

(a) $H(X,Y)$

(b) $H(X)$

(c) $H(Y)$

(d) $H(X|Y)$

(e) $H(Y|X)$

(f) $I(X;Y)$

Problem 3. Compute the capacities of each of the communication channels whose transition probability matrices are specified below.

(a)

$$Q = \begin{bmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix}$$

(b)

$$Q = \begin{bmatrix} 0 & 1/5 & 4/5 & 0 & 0 \\ 2/3 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 1. \end{bmatrix}$$

(c)

$$Q = \begin{bmatrix} 1/3 & 2/3 & 0 \\ 0 & 2/3 & 1/3. \end{bmatrix}$$

(d)

$$Q = \begin{bmatrix} 1/3 & 2/3 \\ 1/3 & 2/3 \end{bmatrix}$$

Problem 4 (Blahut-Arimoto algorithm)**.**

(a) Create a `MATLAB` function `capacity` which calculates the capacity $C = \max_p I(p,Q)$ and the corresponding capacity-achieving input pmf p^* using Blahut-Arimoto algorithm.

The function takes two inputs: (1) the channel transition probability matrix $Q(y|x)$ and (2) the initial guess of the pmf $p_0(x)$.

Define a sequence $p_r(x)$, $r = 0, 1, \ldots$ according to the following iterative prescription

$$p_{r+1}(x) = \frac{p_r(x)\, c_r(x)}{\sum\limits_x p_r(x)\, c_r(x)},$$

where

$$\log c_r(x) = \sum_y Q(y|x) \log \frac{Q(y|x)}{q_r(y)} \tag{4.1}$$

and

$$q_r(y) = \sum_x p_r(x) Q(y|x).$$

After several iterations, the pmf $p_r(x)$ will converge to the capacity-achieving one. In fact,

$$\log \left(\sum_x p_r(x) c_r(x) \right) \le C \le \log \left(\max_x c_r(x) \right). \tag{4.2}$$

So, we can use (4.2) to control the accuracy of our results.

(b) Check your answers in Problem 3 using the Blahut-Arimoto algorithm.

Problem 1: H and I

Wednesday, October 02, 2013 10:19 AM

First, we need to find the unknown constant c.
The given description for the joint pmf can be expressed using the joint pmf matrix as

$$P_{X,Y} = \begin{array}{c|cc} x \backslash y & 2 & 4 \\ \hline 1 & 3c & 5c \\ 3 & 5c & 7c \end{array}$$

Recall that $\sum_x \sum_y p_{X,Y}(x,y) = 1.$

Here, we have

$$3c + 5c + 5c + 7c = 1$$
$$20c = 1$$
$$c = \frac{1}{20}.$$

a) $H(X,Y) = H\left(\left[\frac{3}{20} \ \frac{1}{4} \ \frac{1}{4} \ \frac{7}{20}\right]\right) = -\frac{3}{20}\log_2\frac{3}{20} - \frac{2}{4}\log_2\frac{1}{4} - \frac{7}{20}\log_2\frac{7}{20}$

≈ 1.9406 bits.

To find $H(X)$ and $H(Y)$, we need p_X and p_Y, respectively.
These can be found from the sums along the rows and columns of $P_{X,Y}$.

$$\begin{array}{c|cc|l} x \backslash y & 2 & 4 & \\ \hline 1 & 3c & 5c & \rightarrow 8c = \frac{8}{20} = \frac{2}{5} \\ 3 & 5c & 7c & \rightarrow 12c = \frac{12}{20} = \frac{3}{5} \\ \hline & \downarrow & \downarrow & \\ & 8c & 12c & \\ & \parallel & \parallel & \\ & 2/5 & 3/5 & \end{array}$$

b) $H(X) = H\left(\left[\frac{2}{5} \quad \frac{3}{5}\right]\right) \approx 0.9710$

c) $H(Y) = H\left(\left[\frac{2}{5} \quad \frac{3}{5}\right]\right) \approx 0.9710$

d) $H(X|Y) = H(X,Y) - H(Y) \approx 0.9697$

e) $H(Y|X) = H(X,Y) - H(X) \approx 0.9697$

f) $I(X;Y) = H(X) + H(Y) - H(X,Y) \approx 0.0013$

Problem 2: H and I

Wednesday, October 02, 2013 10:19 AM

First, we need to find the unknown constant β

The given description for the joint pmf can be expressed using the joint pmf matrix as

$$\mathbf{P}_{X,Y} = \begin{array}{c|cc} x \backslash y & 1 & 3 \\ \hline 3 & 1/15 & 4/15 \\ 4 & 2/15 & \beta \end{array}$$

Recall that $\sum_x \sum_y p_{X,Y}(x,y) = 1.$

Here, we have

$$\frac{1}{15} + \frac{4}{15} + \frac{2}{15} + \beta = 1$$

$$\beta = 1 - \frac{7}{15} = \frac{8}{15}.$$

$$\mathbf{P}_{X,Y} = \begin{array}{c|cc|l} x \backslash y & 1 & 3 & \\ \hline 3 & 1/15 & 4/15 & \rightarrow 5/15 = 1/3 \\ 4 & 2/15 & 8/15 & \rightarrow 10/15 = 2/3 \\ \hline & \downarrow & \downarrow & \\ & 3/15 & 12/15 & \\ & \| & \| & \\ & 1/5 & 4/5 & \end{array}$$

Notice that $(\mathbf{p}_X)^T (\mathbf{p}_Y) = \mathbf{P}_{X,Y}$; that is $p_{X,Y}(x,y) = p_X(x)\,p_Y(y)$ for all pair (x,y).

Therefore, $X \perp\!\!\!\perp Y$.

a) $H(X,Y) \overset{X \perp\!\!\!\perp Y}{=} H(X) + H(Y) \approx 1.6402$

b) $H(X) \overset{X \perp\!\!\!\perp Y}{=} H([\,1/3 \quad 2/3\,]) \approx 0.9183$

$X \perp\!\!\!\perp Y$

c) $H(Y) = H\left(\left[\frac{1}{5} \quad \frac{4}{5}\right]\right) \approx 0.7219$

d) $H(X|Y) = H(X) \approx 0.9183$

$\uparrow$ $X \perp\!\!\!\perp Y$ $\downarrow$

e) $H(Y|X) = H(Y) \approx 0.7219$

f) $I(X;Y) = 0$ because $X \perp\!\!\!\perp Y$

Problem 3: C

Wednesday, October 02, 2013 10:51 AM

(a) Symmetric Channel $\Rightarrow C = \log_2 |S_Y| - H(\underline{r}) = \log_2 2 - H([\tfrac{1}{3}\ \ \tfrac{2}{3}])$

$$= 1 - 0.9183 \approx 0.0817$$

(b) Note that this is a noisy channel with non overlapping outputs.

So, $C = \log_2 |S_X| = \log_2 3 \approx 1.5850$.

(c) $I(X;Y) = \underbrace{H(Y)} - \underbrace{H(Y|X)}$

$\hookrightarrow = H([1/3\ \ 2/3])$ ← all rows of Q have the same collection of conditional probabilities.

$$\underline{q} = \underline{p}Q = \begin{bmatrix} p_0 & 1-p_0 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 0 \\ 0 & 2/3 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3}p_0 & \frac{2}{3}p_0 + \frac{2}{3}(1-p_0) & \frac{1}{3}(1-p_0) \end{bmatrix} = \begin{bmatrix} \frac{1}{3}p_0 & \frac{2}{3} & \frac{1}{3}(1-p_0) \end{bmatrix}$$

Because $H(Y|X)$ is fixed, we need to maximize $H(Y)$.

To do this, note that

$$H(Y) = -\frac{1}{3}p_0 \log_2 \frac{1}{3}p_0 - \frac{2}{3}\log_2\frac{2}{3} - \frac{1}{3}(1-p_0)\log_2\frac{1}{3}(1-p_0).$$

$$(\ln 2)H(Y) = -\frac{1}{3}p_0 \ln\left(\frac{1}{3}p_0\right) - \frac{2}{3}\ln\frac{2}{3} - \frac{1}{3}(1-p_0)\ln\left(\frac{1}{3}(1-p_0)\right).$$

$$\Big\downarrow \frac{d}{dp_0}$$

$$= -\frac{1}{3}\left(\cancel{1} + \ln\left(\frac{1}{3}p_0\right)\right) - \left(-\frac{1}{3}\right)\left(\cancel{1} + \ln\left(1 + \frac{1}{3}(1-p_0)\right)\right)$$

$$= \frac{1}{3}\left(\ln\left(\frac{\cancel{1/3}(1-p_0)}{\cancel{1/3}\,p_0}\right)\right) = \frac{1}{3}\ln\left(\frac{1-p_0}{p_0}\right) = \frac{1}{3}\ln\left(\underbrace{\frac{1}{p_0} - 1}\right)$$

↑ As p_0 increases from 0^+ to 1^-,
$\frac{1}{p_0} - 1$ decreases from $+\infty$ to 0^+
$\ln\left(\frac{1}{p_0} - 1\right)$ decreases from $+\infty$ to $-\infty$.

Note: $\frac{d}{dx} f(x)\ln f(x) = f'(x)\ln f(x) + \cancel{f(x)}\frac{1}{\cancel{f(x)}}f'(x)$

$= \left(f'(x)\right)\left(1 + \ln f(x)\right)$

The derivative is 0 at $p_0 = \frac{1}{2}$.

$\frac{1}{3}\ln\left(\frac{1}{p_0}-1\right)=0$

For $p_0 < \frac{1}{2}$, the derivative is >0;

For $p_0 > \frac{1}{2}$, " " <0.

So, $p_0 = \frac{1}{2}$ is the global maximum.

When $p_0 = 1/2$,

$$H(Y) = H\left(\left[\frac{1}{6} \;\; \frac{2}{3} \;\; \frac{1}{6}\right]\right).$$

So,
$$\begin{aligned} C &= H\left(\left[\frac{1}{6} \;\; \frac{2}{3} \;\; \frac{1}{6}\right]\right) - H\left(\left[1/3 \;\; 2/3\right]\right) \\ &= -\frac{2}{6}\log_2\frac{1}{6} - \frac{2}{3}\cancel{\log_2}\frac{2}{3} + \frac{1}{3}\log_2\frac{1}{3} + \frac{2}{3}\cancel{\log_2}\frac{2}{3} \\ &= \frac{1}{3}\log_2\frac{6}{3} = \frac{1}{3}. \end{aligned}$$

(d) The rows of Q are the same. So, $q(y) = Q(y|x)$ which implies $X \perp\!\!\!\perp Y$.

Therefore, $I(X;Y)=0$ for any input distribution.

Hence, $C = 0$.

ECS 452: Digital Communication Systems **2013/1**

HW 5 — Due: Not Due

Lecturer: Prapun Suksompong, Ph.D.

Instructions

(a) Have fun!

Problem 1. Consider a standard rectangular 8-ary constellation shown in Figure 1. As usual, it is derived from the waveform models whose noise process is additive white Gaussian noise (AWGN) with PSD $\frac{N_0}{2} = 3$. The constellation is centered at the origin (so that the average E_s is minimized.) The vertical distances and horizontal distances between any adjacent points are all $d = 1$. Minimum-distance detector is used.

(a) Find the (1,2) element in the Q matrix. (This is the probability that the detector output is $\hat{W} = 2$ given that the actual intended message is $W = 1$.)

(b) Find the (3,5) element in the Q matrix.

(c) Find the value of $\frac{E_b}{N_0}$ for this constellation under the above description of noise. Assume equiprobable messages.

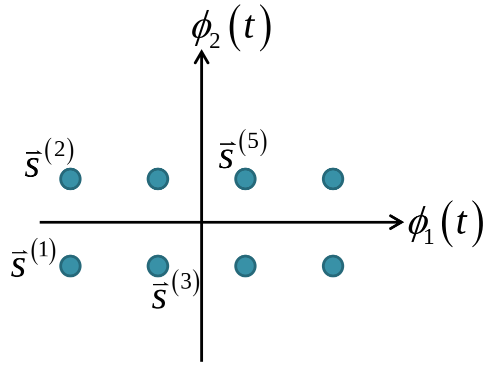


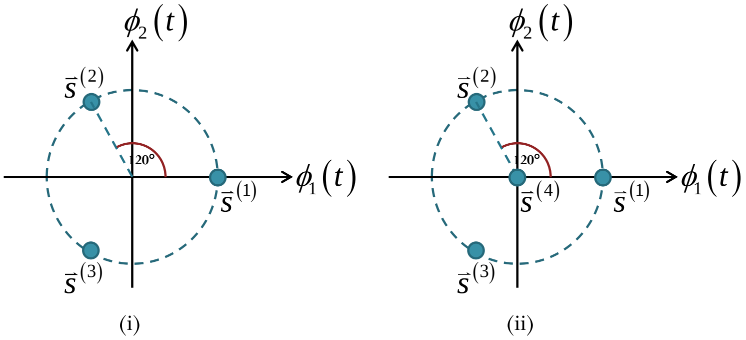
Figure 5.1: Constellations for Problem 1.

Problem 2. Often, we have to work with constellation that is difficult to derive the Q matrix (because the integrations involved are difficult. It's best to try to reduce the number of calculations that are needed.

In class, we have seen that, for QPSK, even though there are $4^2 = 16$ possible elements in the matrix Q, we only have to identify three elements in there. Here, consider the constellations in Figure 5.2.i and Figure 5.2.ii. Let's suppose that you have already calculated some elements of their Q matrices to be

$$Q = \left[\begin{array}{c|c|c} & 0.30 & \\ \hline & & \\ \hline & & \end{array}\right] \text{ and } Q = \left[\begin{array}{c|c|c|c} 0.41 & 0.29 & & \\ \hline & & & \\ \hline & & & \\ \hline & & 0.33 & \end{array}\right],$$

respectively.



(i) (ii)

Figure 5.2: Constellations for Problem 2.

(a) Find the values of the rest of the elements. Assume minimum-distance (maximum-likelihood) decoder and AWGN channel.

(b) Find the (overall average) probability of (detection) error for each constellation. Assume that the points are equally likely.

Problem 3. Consider the vector channel derived from waveform channels under AWGN with PSD $\frac{N_0}{2}$. We consider two digital modulation: BPSK and QPSK. The detector at the receiver uses minimum-distance detection.

(a) Derive the formula and then plot the capacity of BPSK as a function of $\frac{E_b}{N_0}$.

(b) Derive the formula and then plot the capacity of QPSK as a function of $\frac{E_b}{N_0}$.

Problem 4. A linear block code has a generator matrix

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

.

(a) List all codewords for this code.

(b) Determine a suitable parity check matrix H.

(c) Check that $GH^T = 0$

(d) Find the minimum distance of this code.

Problem 5. A Hamming code has the parity check matrix given by

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

(a) What is the number of parity bits used in this code?

(b) Find the corresponding generator matrix.

(c) The following information bits are to be encoded using the Hamming code above:

001110111010

(i) How should the bits be split into blocks? In particular, what is the length of each block and how many blocks are used?

(ii) Find the corresponding codewords

(d) Some more information bits were generated. They were encoded using the Hamming code above. Suppose the received bits are

101101110000010100001

(i) How should the received bits be split into blocks (received vectors)? In particular, what is the length of each block and how many blocks are there?

(ii) Locate and correct all errors.

Q1

Friday, October 11, 2013 9:28 AM

When the AWGN has PSD $= \frac{N_0}{2}$, we know that the noise in vector form will have $\sigma^2 = \frac{N_0}{2}$ in each dimension.

To solve part (a) and (b), we first sketch the decision regions. Here, the boundaries of the regions can be easily found because the detector use minimum-distance detection.

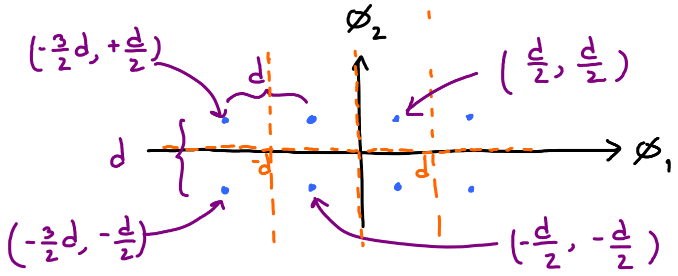
(Earlier, we saw that this detector is optimal (same as the MAP detector) when

(1) the vector channel model is additive Gaussian noise

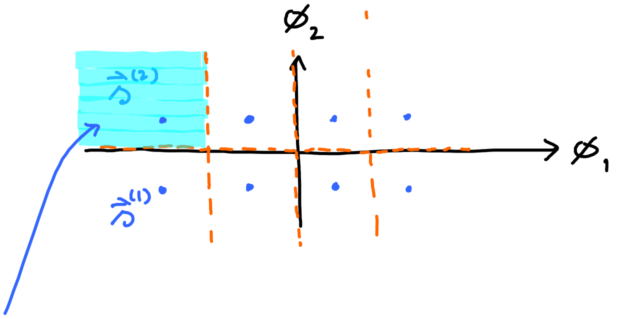
and

(2) the messages are equally likely.)

The boundaries are simply the perpendicular bisectors of the lines connecting two (vertically or horizontally) adjacent signal points.



(a) $Q_{2,2} = P[\hat{w} = 2 \mid w = 1]$



To be detected as $\vec{s}^{(2)}$, the received vector must be in D_2

the decision region for $\vec{s}^{(2)}$.

The first component of $\vec{N}$ (Noise in the 1st dimension)

$$Q_{12} = P\left[-\infty < N_1 \le \frac{d}{2}\right] P\left[\frac{d}{2} \le N_2 < \infty\right]$$

The second component of $\vec{N}$ (Noise in the 2nd dimension)

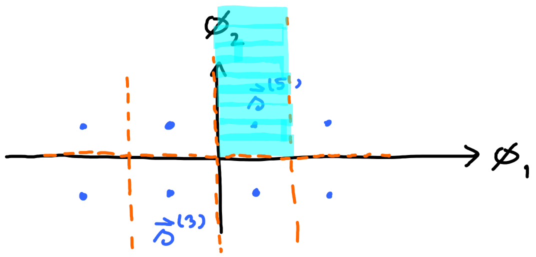
(Noise in the 2nd dimension)

$$= \left(Q\left(\frac{-\infty}{\sigma}\right) - Q\left(\frac{d}{2\sigma}\right)\right)\left(Q\left(\frac{d}{2\sigma}\right) - Q\left(\frac{\infty}{\sigma}\right)\right) = \left(1 - Q\left(\frac{d}{2\sigma}\right)\right)\left(Q\left(\frac{d}{2\sigma}\right)\right)$$

Here, $\sigma^2 = \frac{N_0}{2}$. So $\sigma = \sqrt{\frac{N_0}{2}}$ and $\frac{d}{2\sigma} = \frac{d}{2\times\sqrt{\frac{N_0}{2}}} = \frac{1}{2\sqrt{3}}$.

Therefore, $Q_{12} = \left(1 - Q\left(\frac{1}{2\sqrt{3}}\right)\right)\left(Q\left(\frac{1}{2\sqrt{3}}\right)\right) \approx 0.2371$

(b)



$$Q_{3,5} = P[\hat{W} = 5 \mid W = 3]$$

$$= P\left[\frac{d}{2} \leq N_1 \leq \frac{3d}{2}\right] P\left[+\frac{d}{2} \leq N_2 < \infty\right]$$

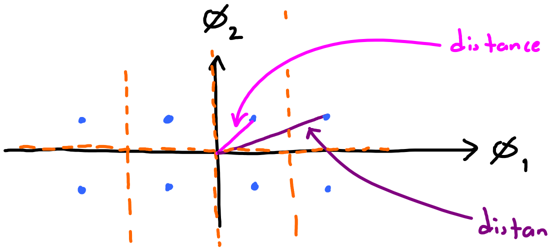
$$= \left(Q\left(\frac{d}{2\sigma}\right) - Q\left(\frac{3d}{2\sigma}\right)\right)\left(Q\left(+\frac{d}{2\sigma}\right) - \cancelto{0}{Q\left(\frac{\infty}{\sigma}\right)}\right)$$

$$= \left(Q\left(\frac{d}{2\sigma}\right) - Q\left(\frac{3d}{2\sigma}\right)\right)\left(\cancel{1-}\,Q\left(\frac{d}{2\sigma}\right)\right)$$

$$= \left(Q\left(\frac{1}{2\sqrt{3}}\right) - Q\left(\frac{\sqrt{3}}{2}\right)\right)\left(\cancel{1-}\,Q\left(\frac{1}{2\sqrt{3}}\right)\right)$$

$\approx$ [illegible] 0.0746

(c)



distance to origin $= \sqrt{\left(\frac{d}{2}\right)^2 + \left(\frac{d}{2}\right)^2}$

$$= \sqrt{\frac{2d^2}{4}} = \frac{d}{\sqrt{2}}$$

distance to origin $= \sqrt{\left(\frac{3d}{2}\right)^2 + \left(\frac{d}{2}\right)^2}$

$$= \sqrt{\frac{10d^2}{4}} = d\sqrt{\frac{5}{2}}$$

$$E_s = \frac{1}{8} \times \left(4 \times \left(\frac{d}{\sqrt{2}}\right)^2 + 4 \times \left(d\sqrt{\frac{5}{2}}\right)^2\right) = \frac{1}{2}\left(\frac{d^2}{2} + \frac{5d^2}{2}\right) = \frac{1}{2} \times 3d^2 = \frac{3}{2}d^2$$

$$E_b = \frac{E_s}{\log_2 M} = \frac{\frac{3}{2}d^2}{\log_2 8} = \frac{\frac{3}{2}d^2}{3} = \frac{1}{2}d^2 \underset{d=1}{\overset{\uparrow}{=}} \frac{1}{2}$$

$$\frac{E_b}{N_0} = \frac{1/2}{2\times 3} = \frac{1}{12}.$$

Q2

Friday, October 11, 2013 11:43 AM

(a) (i) Step ①

Same arrangement of point vs. decision region.

$$Q = \begin{bmatrix} & 0.3 & 0.3 \\ 0.3 & & 0.3 \\ 0.3 & 0.3 & \end{bmatrix}$$

Step ②

$1-0.3-0\ 3 = 0.4$

$$Q = \begin{bmatrix} & 0.3 & 0.3 \\ 0.3 & & 0.3 \\ 0.3 & 0.3 & \end{bmatrix}$$

Conclusion : $Q = \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.3 & 0.4 & 0.3 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}$

(ii)

$$Q = \begin{bmatrix} 0.41 & 0.29 & 0.29 & 0.01 \\ 0.29 & 0\ 41 & 0.29 & 0.01 \\ 0.29 & 0.29 & 0.41 & 0.01 \\ 0.33 & 0.33 & 0.33 & 0.01 \end{bmatrix}$$

(b) $P(\mathcal{E}) = P[\hat{W} \neq W] = 1 - P[\hat{W} = W] = 1 - \sum_i P[\hat{W} = W | W = i] P[W = i]$

$= 1 - \sum_i P[\hat{W} = i | W = i] \underbrace{\frac{1}{M}}_{\text{equally likely messages}} = 1 - \sum_i Q_{i,i} \frac{1}{M} = 1 - \frac{1}{M} \operatorname{tr}(Q).$

$\operatorname{tr}(A)$ = the sum of the elements on the main diagonal of A.

(i) $P(\mathcal{E}) = 1 - \frac{1}{3}(0.4 + 0\ 4 + 0.4) = 0.6$

(ii) $P(\mathcal{E}) = 1 - \frac{1}{4}(3 \times 0.41 + 0.01) = 0.69$

(a) BPSK waveform channel gives binary symmetric channel with crossover probability $p = Q\left(\frac{d}{2\sigma}\right) = Q\left(\sqrt{\frac{4E_b}{2N_0}}\right) = Q\left(\sqrt{2\frac{E_b}{N_0}}\right)$

$E_s = \left(\frac{d}{2}\right)^2 = \frac{d^2}{4}$

$E_b = \frac{E_s}{\log_2 M} = \frac{d^2/4}{\log_2 2} = \frac{d^2}{4}$

$d = \sqrt{4E_b}$

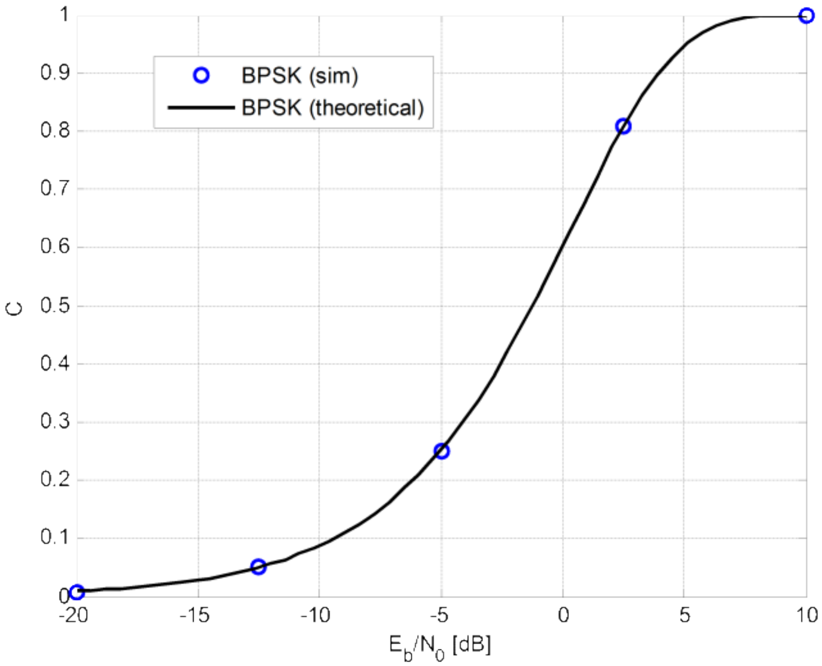
$\sigma^2 = \frac{N_0}{2}$

$\sigma = \sqrt{\frac{N_0}{2}}$

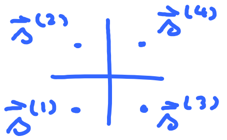
$2\sigma = \sqrt{2N_0}$

The capacity of BSC is given by $1 - H(p) =$

$= 1 + (p\log_2 p + (1-p)\log_2(1-p))$ where $p = Q\left(\sqrt{2\frac{E_b}{N_0}}\right)$.



(b) In class, we have shown that the Q matrix of standard rectangular quaternary QAM is given by

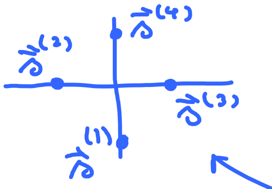


$$Q = \begin{bmatrix} (1-q)^2 & q(1-q) & q(1-q) & q^2 \\ q(1-q) & (1-q)^2 & q^2 & q(1-q) \\ q(1-q) & q^2 & (1-q)^2 & q(1-q) \\ q^2 & q(1-q) & q(1-q) & (1-q)^2 \end{bmatrix}$$ where $q = Q\left(\frac{d}{2\sigma}\right)$

This is a symmetric channel and the corresponding capacity is

$$C = \log_2 |S_Y| - H(\vec{r}) = \log_2 4 - H\left(\left[(1-q)^2 \quad q(1-q) \quad q(1-q) \quad q^2\right]\right)$$

To get the constellation for QPSK, we simply rotate the constellation above by 45°.



Because the relative positions of the points in the constellation are the same, we have the same Q matrix (if the points are numbered as shown)

So, the capacity is the same as what we found above.

Note also that $E_s = \left(\frac{d}{\sqrt{2}}\right)^2 = \frac{d^2}{2}$ and $E_b = \frac{E_s}{\log_2 M} = \frac{E_s}{\log_2 4} = \frac{d^2/2}{2} = \frac{d^2}{4}$

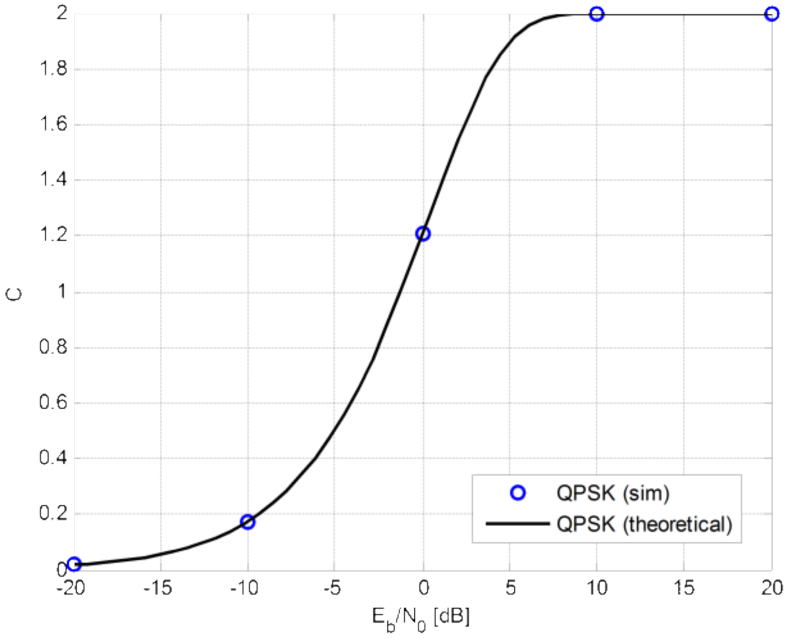
$$d = \sqrt{4E_b}$$

$$\frac{d}{2\sigma} = \sqrt{\frac{4E_b}{2N_0}} = \sqrt{2\frac{E_b}{N_0}}.$$

Therefore,

$$C = 2 + (1-q)^2 \log_2 (1-q)^2 + 2q(1-q)\log_2 q(1-q) + q^2 \log_2 q^2$$

where $q = Q\left(\sqrt{2\frac{E_b}{N_0}}\right)$


2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
C
-20
-15
-10
-5
0
5
10
15
20
E_b/N_0 [dB]
QPSK (sim)
QPSK (theoretical)

Q4

Friday, October 11, 2013 1:35 PM

$$G = \left[\underbrace{\begin{matrix}1&0&0&0\\0&1&1&1\end{matrix}}_{P}\ \vdots\ \underbrace{\begin{matrix}1&0\\0&1\end{matrix}}_{I}\right]$$

(a) Any codeword is of the form $\underline{x} = \underline{b}G$.

Here, G has $k = 2$ rows; so, the length of $\underline{b}$ must also be $k = 2$.

There fore, there are $2^k = 2^2 = 4$ codewords.

$\underline{b}$	$\underline{x} = \underline{b}G$
00	000000
01	011101
10	100010
11	111111

(b) $$H = [I \vdots -P^T] = \begin{bmatrix}1&0&0&0&1&0\\0&1&0&0&0&1\\0&0&1&0&0&1\\0&0&0&1&0&1\end{bmatrix}$$

the negative sign does not matter in GF(2)

(c)

add these two rows

$$GH^T = \begin{bmatrix}1&0&0&0&1&0\\0&1&1&1&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&1&1\end{bmatrix} = \begin{bmatrix}0&0&0&0\\0&0&0&0\end{bmatrix} \checkmark$$

(d) $d_{min} = 2$ ← There are $\binom{4}{2} = 6$ pairs of codewords.

The minimum Hamming distance among these pairs is 2.

Q5

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$$H = \left[\overbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}^{I} \;\; \underbrace{\begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{matrix}}_{P}\right]$$

(a) Matrix H should be $(n-k) \times n$.

So, $n = 7$,

$n - k = 3$,

$k = n - 3 = 7 - 3 = 4$.

(b)

$$G = [\underset{\uparrow}{-}P^T \mid I] = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

the negative sign does not matter in GF(2)

(c)

(i) From (a), we know that $k = 4$. Therefore, the information bits should be divided into blocks of 4 bits.
12 bits are given. So, there should be $\frac{12}{4} = 3$ blocks.

(ii)

$\underline{b}$	$\underline{x} = \underline{b}\,G$
0 0 1 1	0 0 1 0 0 1 1
1 0 1 1	0 1 0 1 0 1 1
1 0 1 0	1 0 1 1 0 1 0

(d)

(i) From (a), we know that $n = 7$. Therefore, the received bits should be divided into blocks of 7 bits
21 bits are given. So, there should be $\frac{21}{7} = 3$ blocks.

(ii)

$\underline{y}$	$\underline{s} = \underline{y}H^T$		Error position	Corrected $\underline{x}$
1 0 1 1 0 1 1	1 1 1	→ 7th column of H	7	1 0 1 1 0 1 0
1 0 0 0 0 0 1	0 1 1	→ 4th column of H	4	1 0 0 1 0 0 1
0 1 0 0 0 0 1	1 0 1	→ 5th column of H	5	$010\underbrace{0101}_{\hat{\underline{b}}}$